Mathematics

HP COMPUTER CURRICULUM

Linear Equations & Systems

STUDENT LAB BOOK

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Hewlett-Packard
Computer Curriculum Series

mathematics STUDENT BOOK

linear equations & systems

by E.A. Herber
Golden High School
Golden, Colorado

edited by Jean Danver and Christine Doerr
Hewlett-Packard

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This material is designed to be used with any Hewlett-Packard system with the BASIC program language such as the 9830A, Educational BASIC, and the 2000 and 3000 series systems.

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TABLE OF CONTENTS

INTRODUCTION

This Mathematics Student Book was written to enrich your study of mathematics by showing you how to use a computer as a problem solving tool to model mathematical concepts. The computer is particularly helpful in quickly performing the repetitious steps of logarithms, thus making mathematical investigations easier and more exciting. You will write computer programs which will help you to understand the major concepts involved in the study of a particular mathematical area. If you become more involved in investigating the laws of mathematics, these Books will have achieved their aim.

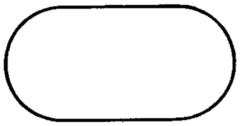
To use the Lab Book for Linear Equations, you will need the following: First, you should have one year's background in algebra. Second, the Lab Book assumes that you already know how to write a program in the BASIC programming language, and that you understand programming techniques for inputting data, performing algebraic operations, designating variables, assigning values to variables, looping, and printing results. If you do not have this background, you will want to study BASIC before attempting this material. Consult the BASIC manual for the computer you use. Last, in order to complete the exercises in this Lab Book, you will need to have access to a computer for at least two hours per week. If more time is available, you may be able to experiment further on your own, either to improve your program or to investigate other areas of mathematics that interest you.

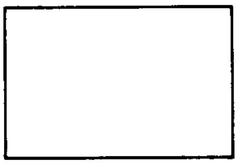
Each section of this book is organized in the same way. First, the mathematical concepts needed to complete the exercises are reviewed. References are listed at the end of each section in case you want to study these concepts in greater detail. Next, each exercise is presented. Finally, an approach is suggested in the Problem Analysis and a flowchart is included to illustrate this approach. The suggested approach was chosen because it brings out the concepts which are being stressed, but the program can sometimes be written more efficiently. Once you have completed the exercise by following the logic in the flowchart, you are encouraged to rewrite the program using more sophisticated programming techniques. You might also want to impose more conditions on the problem to make it more interesting to solve.

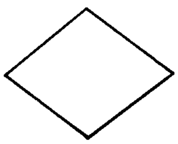
There is no one "right" way to solve a problem by programming. Experiment and learn as you go. You'll find you are learning something new each time, both about your subject matter and about using the computer to solve problems.

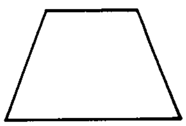
LIST OF SYMBOLS

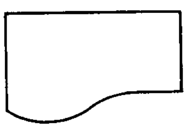
FLOW CHART SYMBOLS

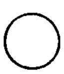

Start or Stop


Defines a process


Represents a decision point


Represents computer input


Represents computer output


Used to connect one part of a flow chart continued at some other place

ALGEBRAIC NOTATION AND EQUIVALENT *BASIC* LANGUAGE SYMBOLS

Algebraic Notation	BASIC Notation	Meaning
+	+	Addition
-	-	Subtraction
$\cdot$ or x	*	Multiplication
÷ or /	/	Division
$\sqrt{x}$	SQR(x)	Square root of x
$\vert y \vert$	ABS(y)	Absolute value of y
[x]	INT(x)	Greatest integer less than or equal to x
=	=	Equals
$\neq$	# or <>	Does not equal
<	<	Less than
>	>	Greater than
$\leqslant$	<=	Less than or equal to
$\geqslant$	>=	Greater than or equal to
$\leftarrow$	=	Replaced by
( ) or [ ]	( ) or [ ]	Inclusive brackets or parentheses
A_i	A(I)	Subscripted variable
$A_{i,j}$	A(I,J)	Double subscripted variable
(None)	RND(X)	Assign a random number to the variable X

NUMBER SET DESIGNATIONS

N - Natural number set

W - Whole number set

I - Set of integers

Q - Rational number set

Z - Irrational number set

R - Real number set

LINEAR EQUATIONS

General Form

A *linear equation* defines a set of points whose graph is a straight line. The *general form* of a linear equation is

$$Ax + By + C = 0, \tag{1}$$

with $A \neq 0$ or $B \neq 0$. If $A=0$ and $B \neq 0$, the line defined by the equation is horizontal. If $B=0$ and $A \neq 0$, the line is vertical.

Linear equations are used continuously in the study of science and mathematics. For example, the relationship between the readings on fahrenheit and centigrade thermometers is defined by a linear equation, $F = 9/5\ C + 32°$ (or, in the general form, $5F - 9C - 32° = 0$).

Slope-Intercept Form

The general form of a linear equation can be shown to be equivalent to the *slope-intercept form*

$$y = mx + b, \tag{2}$$

where $m = \frac{-A}{B}$ and $b = \frac{-C}{B}$.

We'll define m to be the slope of the line, determined by dividing the change in y (called Δy) by the change in x (Δx) as we move from one point to another on the line. In other words

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

for two points (x_2, y_2) (x_1, y_1) on the line.

Let's graph the line defined by (2), assuming that m and b are both greater than 0.

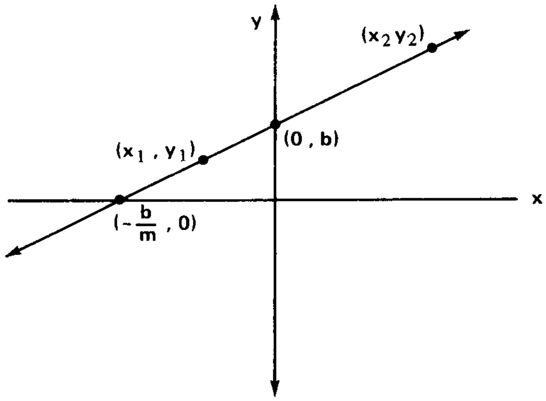


Figure 1. Graph of y = mx + b

The point where the line crosses the y axis is called the y-intercept, in this case (o,b). So you can see why we call the equation y=mx+b the slope-intercept form.

If $x_2 - x_1 = 0$, then the slope is undefined and the graph of the equation is a vertical line. If $y_2 - y_1 = 0$, then the slope is zero and the graph is a horizontal line.

A line doesn't necessarily have a positive slope. Take the equation 2x + 3y -6 = 0. To put this in slope-intercept form, we solve for y, finding that y = - 2/3 x + 2. From our definition, we assume that the slope equals - 2/3. We can verify that by graphing the function and calculating the slope.

x	y
-3	4
0	2
3	0

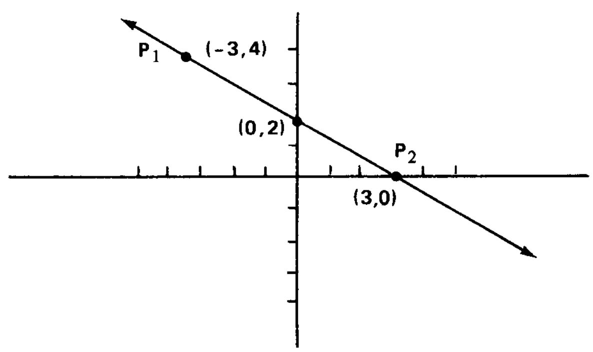


Figure 2. Graph of $y = -2/3\ x + 2$

From the coordinates of the points P_1 and P_2 we calculate the slope,

$$m = \frac{\Delta y}{\Delta x} = \frac{y_1 - y_2}{x_1 - x_2} = \frac{4 - 0}{-3 - 3} = -\frac{2}{3}.$$

Point-Slope Form

The *point-slope form* for a linear equation, also equivalent to the general form, is

$$y - y_1 = m(x - x_1), \tag{3}$$

where m is the slope, (x_1, y_1) is a given point and (x,y) is any other point on the line defined by the equation.

Two Point Form

A linear equation can also be expressed by using any two points, (x_1, y_1) and (x_2, y_2), $x_1 \neq x_2$, in what is known as the *two-point form* of the equation:

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \tag{4}$$

Exercise 1 — Converting the General Form to Slope-Intercept Form

Write a computer program that will receive a linear equation of the general form, $Ax + By + C = 0$, $A \neq 0$ or $B \neq 0$, and output the slope, the y-intercept and the slope-intercept form of the equation. Apply your program to the following equations.

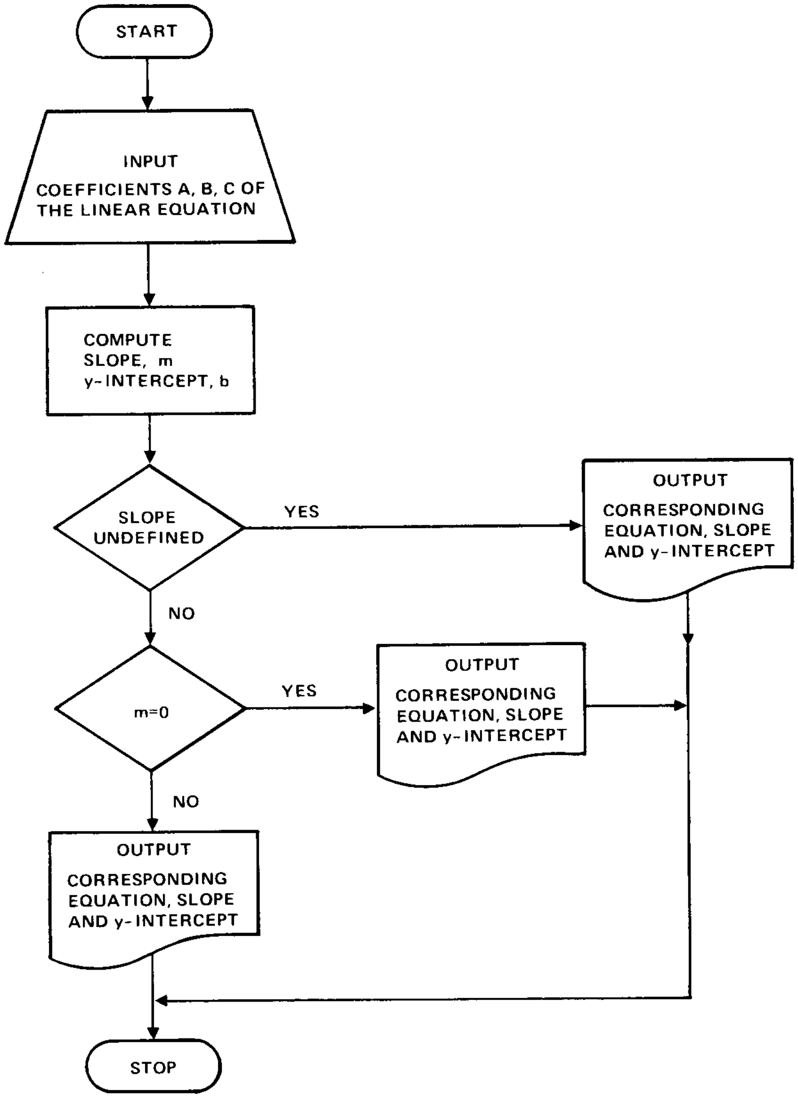
a. $3x - 2y + 7 = 0$

b. $.23x + .1y - .6 = 0$

c. $8x - 2y = 7$

d. $3x + 4 = 0$

e. $2y - 5 = 0$


START
INPUT
COEFFICIENTS A, B, C OF THE LINEAR EQUATION
COMPUTE
SLOPE, m
y-INTERCEPT, b
SLOPE UNDEFINED
YES
OUTPUT
CORRESPONDING EQUATION, SLOPE AND y-INTERCEPT
NO
m=0
YES
OUTPUT
CORRESPONDING EQUATION, SLOPE AND y-INTERCEPT
NO
OUTPUT
CORRESPONDING EQUATION, SLOPE AND y-INTERCEPT
STOP

Exercise 2 — Finding the General Form from Characteristics of the Line

Write a computer program that will take a set of data, in any one of the forms discussed below and print an equation, in the general form, of a line that has those characteristics. Apply the program to:

a.	*(2,-1) and (-3.2,7)*	*f.*	*(3.27,-6) and Y-intercept=-5*
b.	*(0,8) and (-6,0)*	*g.*	*m=7/8, y-intercept= 7.3*
c.	*(.8,-2) and (16,-2)*	*h.*	*m=5/2, y-intercept= 0*
d.	*(-1.5,2) and (-1.5,-8)*	*i.*	*m=0, y-intercept = 2*
e.	*(-7.5,-3) and m=-3*		

Problem Analysis

We can find an equation for a line if we have any pair of characteristics as listed is the cases below:

1. Two points on the line
2. A point on the line, slope of the line
3. A point on the line, y-intercept of the line
4. Slope of the line, y-intercept of the line

Consider the first possible case, where we are given two points on a line. Let's look again at the two-point form of the linear equation.

$$y-y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x-x_1)\ , x_2 \neq x_1. \tag{4}$$

We can show that equation (4) is equivalent to

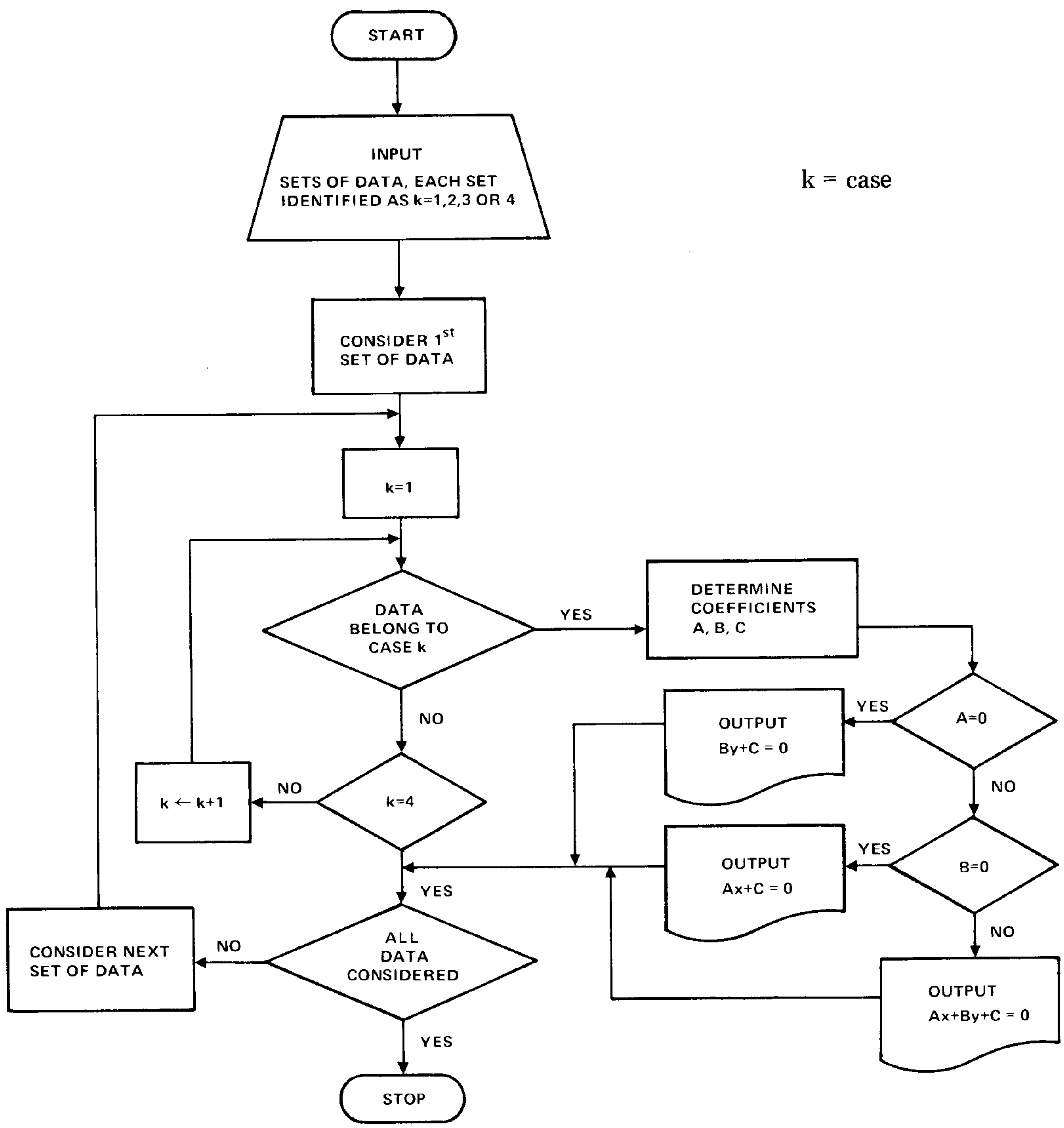
$$(y_1 - y_2)x + (x_2 - x_1)y - y_1 x_2 + x_1 y_2 = 0 \tag{5}$$

which is in the general form with $A = y_1 - y_2$; $B = x_2 - x_1$; and $C = -y_1 x_2 + x_1 y_2$. Thus, to find the general form of the equation from two given points, (x_1, y_1) and (x_2, y_2), we need only substitute values into the expressions for A, B, and C above.

In Case 2 we are given a point on the line and the slope of the line. We'll follow the same procedure we used for the two-point case except our manipulations will involve the point-slope form, $y-y_1 = m(x-x_1)$. This equation simplifies to $mx - y - mx_1 + y_1 = 0$, which is in the general form with $A = m$; $B = -1$; and $C = -mx_1 + y_1$.

In Case 3 we are given a point on the line and the y-intercept of the line. This is equivalent to the case involving two points on the line, since the y-intercept identifies the second point, (0, b).

Case 4 involves the slope and the y-intercept of the line. The slope-intercept form, $y = mx + b$, can be converted into the general form, $-mx + y - b = 0$, with $A = -m$; $B = 1$; and $C = -b$.



k = case

Collinearity of Points

Three or more points are said to be *collinear* if they are points on the same straight line. We can determine if several points are collinear by finding the slope of the line segments joining one point to each of the other points in the set. If all segments have the same slope, the points are collinear.

Exercise 3 — Determining Collinearity

Write a computer program to determine if three or more points in the Cartesian plane are collinear. If the points are collinear, give the slope of the line determined, and identify vertical and horizontal lines.

Apply the program to each of the following sets of points:

a. (-3,-2), (0,0) (6,4) (3,2)

b. (-1,5) (3,5) (4.5,5)

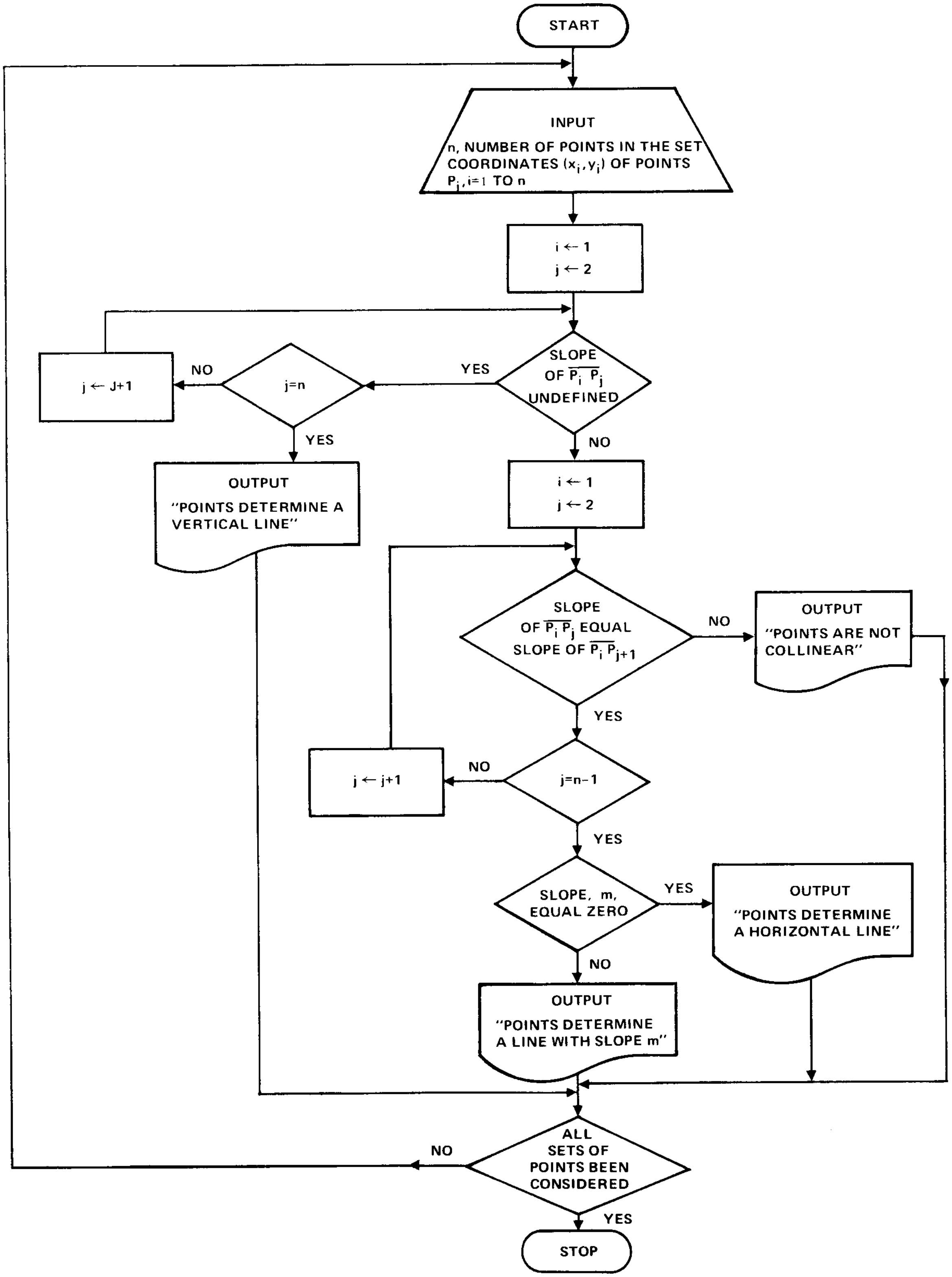
c. (2,-5) (-5,-10) (2,-5) (3,6)

d. (1,2) (-5,-10) (2,-5) (3,6)

e. (2,0) (2,-3.5) (2,8) (2,-7)

Problem Analysis

First, compute the slopes to determine collinearity. If the slopes are all equal to zero, identify the line as horizontal. If the slopes are all undefined, identify the line as vertical. Otherwise, output the slope of the line.


START
INPUT
n, NUMBER OF POINTS IN THE SET
COORDINATES (x_i, y_i) OF POINTS
P_i, i=1 TO n
i ← 1
j ← 2
SLOPE OF $\overline{P_i\ P_j}$ UNDEFINED
YES
j=n
NO
j ← J+1
YES
OUTPUT
"POINTS DETERMINE A VERTICAL LINE"
NO
i ← 1
j ← 2
SLOPE OF $\overline{P_i\ P_j}$ EQUAL SLOPE OF $\overline{P_i\ P_{j+1}}$
NO
OUTPUT
"POINTS ARE NOT COLLINEAR"
YES
j=n−1
NO
j ← j+1
YES
SLOPE, m, EQUAL ZERO
YES
OUTPUT
"POINTS DETERMINE A HORIZONTAL LINE"
NO
OUTPUT
"POINTS DETERMINE A LINE WITH SLOPE m"
ALL SETS OF POINTS BEEN CONSIDERED
NO
YES
STOP

Length and Midpoint of a Line

Figure 3 shows an arbitrary line segment P_1P_2. We've drawn P_1P_3 parallel to the x-axis and P_2P_3 parallel to the y-axis. You can see that P_3 has coordinates (x_2, y_1). The length of P_1P_3 is obviously $x_2 - x_1$, and that of P_2P_3 is $y_2 - y_1$. Therefore, by the Pythagorean Theorem, the length

$$\overline{P_1P_2} = \sqrt{(x_2 - x)^2 + (y_2 - y_1)^2} \tag{6}$$

To determine the coordinates (x,y) of the midpoint, P_m, draw lines from P_m through the midpoints and perpendicular to line segments $\overline{P_1P_3}$ and $\overline{P_2P_3}$. You can see from the figure that

$$x = \frac{x_1 + x_2}{2} \quad \text{and} \quad y = \frac{y_1 + y_2}{2} .$$

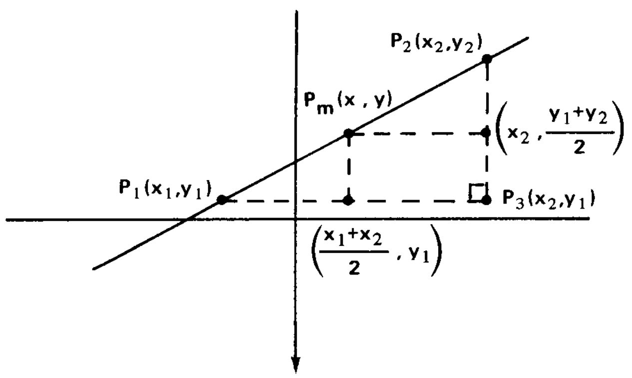


Figure 3. Graph of An Arbitrary Line Segment

Exercise 4 — Finding the Length and Midpoint of a Given Line

Given two points ($P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ in the Cartesian plane, write a program that will determine the length of line segment $\overline{P_1P_2}$ and the coordinates of its midpoint.

Apply your program to the following pairs of points.

a. *(-3,5), (2,-7)*

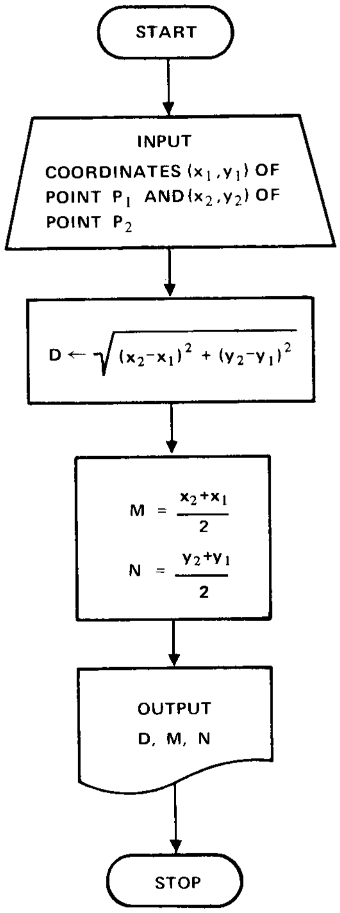
b. *(-7,-1), (8,5)*

c. *(-5,7), (3.5,7)*

d. *(-4,-6.25), (-4.4,8)*

Problem Analysis

Determining the length and midpoint of $\overline{P_1 P_2}$ is a straightforward process as indicated by the flowchart.



D = LENGTH OF LINE CONNECTING P_1 AND P_2

M = x COORDINATE OF MIDPOINT

N = y COORDINATE OF MIDPOINT

Intersecting Lines

Figure 4 shows a line, $\overline{P_1 P_2}$, and its perpendicular bisector, L. To find the linear equation of L we need to know its slope and its y-intercept.

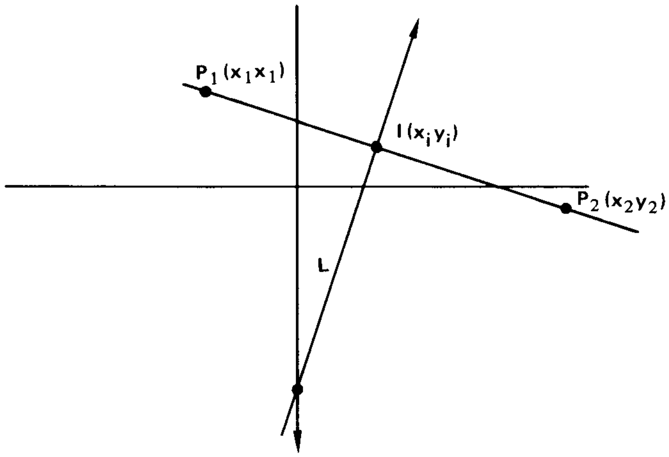


Figure 4. Line $\overline{P_1P_2}$ and its Perpendicular Bisector

First, let's find the slope, m_L. It can be shown that the slope of L is equal to the negative inverse of the slope of $\overline{P_1P_2}$, that is,

$$m_L = \frac{1}{-m} \qquad \text{where } m = \text{slope of } \overline{P_1P_2}.$$

If you are interested in studying the proof of this statement, consult one of the references given at the end of this section. At any rate, we can easily calculate the slope of $\overline{P_1P_2}$ because we are given the coordinates of two points, and that tells us the slope of L.

Now, we need to determine the y-intercept, b, of L. In order to fine b, we will need to determine the intersection point, $I(x_i,y_i)$, of L and $\overline{P_1P_2}$. Remember, the exercise calls for the perpendicular *bisector* of $\overline{P_1P_2}$. Therefore, I is the midpoint of $\overline{P_1P_2}$ and $x_i = (x_2+x_1)/2$ and $y_i = (y_2+y_1)/2$. By substituting these values into the slope-intercept form of the linear equation, we have

$$y_i = m_L\, x_i + b. \tag{7}$$

Since we know the values of y_i, m_L and x_i, we now have the y-intercept, which we'll call b_L. Thus, the linear equation for L is $y = m_Lx + b_L$

Exercise 5 — Equation for the Perpendicular Bisector of a Line

Write a computer program to find the slope-intercept form of the equation of the line which is the perpendicular bisector of a line segment determined by two points P_1 and P_2. Apply your program to the given sets of points.

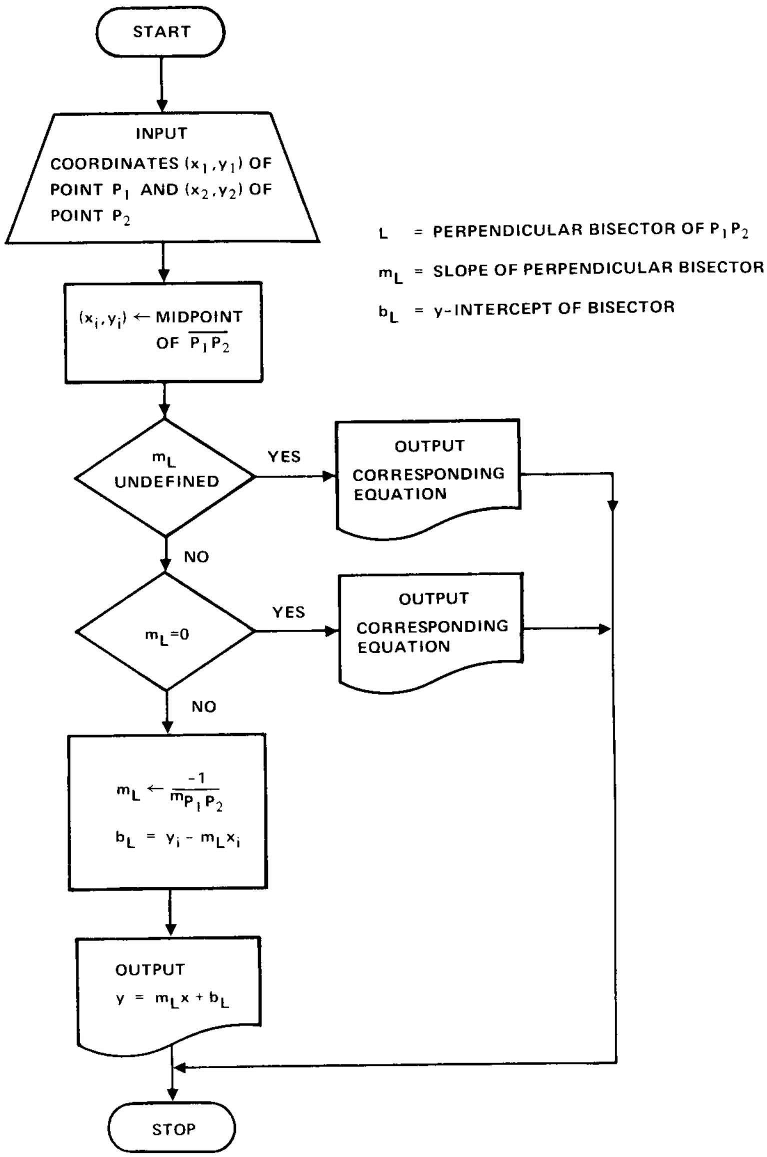
a. *(-7,2); (5,-6)*

b. *(7,0); (7,17)*

c. *(-11.5, -3); (1,-3)*

Problem Analysis

The equation for the perpendicular bisector is found by following the procedure discussed above and outlined in the flowchart.



Another interesting problem involves finding the point of intersection for any two non-parallel lines. In order to do this we need to find the common solution for the equations of the two lines. If there is no common solution, the lines do not intersect. If there is more than one common solution, the equations define what we call co-incident lines rather than intersecting straight lines. The method of determining the common solution is presented in the Problem Analysis, but if you are interested in the details of its development, you should consult one of the suggested reference books.

Exercise 6 — The Intersection of Two Lines

Write a computer program that will find the point of intersection of two lines, given their equations in the general form. Apply the program to:

a. $2x-y+3=0$
$x+4y+2=0$

c. $2x+2y=99$
$x+y-9=0$

b. $1/4x+y=2$
$x/2+3y/5=14$

d. $x+2y-3=0$
$4x+8y=12$

Problem Analysis

Finding the point of intersection of two lines is a problem of finding the common solution of two linear equations, if such a solution exists. Lets assume the two lines L_1 and L_2 being considered have equations $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$, respectively. Further, we assume the graphs of the two equations are as shown below

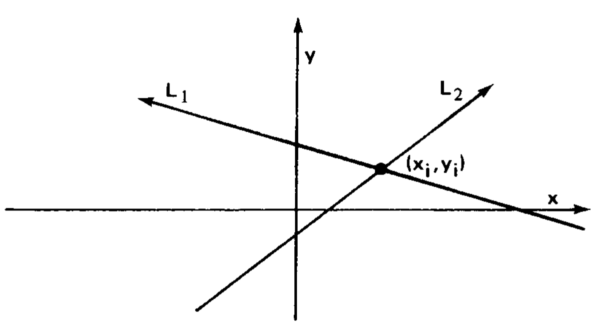


Figure 5. Two Intersecting Lines

To find the point of intersection, (x_i,y_i), we must solve the system below, i.e., find a common solution for the two equations:

$$\begin{cases} A_1x + B_1y + C_1 = 0 \\ A_2x + B_2y + C_2 = 0 \end{cases} \tag{8}$$

It can be shown that the system in (8) is equivalent to:

$$\begin{cases} x = \dfrac{B_1C_2 - B_2C_1}{A_1B_2 - A_2B_1} \\ \\ y = \dfrac{A_2C_1 - A_1C_2}{A_1B_2 - A_2B_1} \end{cases} \tag{9}$$

Therefore, the point of intersection is

$$(x_i,y_i) = \left(\frac{B_1C_2 - B_2C_1}{A_1B_2 - A_2B_1}, \quad \frac{A_2C_1 - A_1C_2}{A_1B_2 - A_2B_1}\right)$$

Let us analyze what the characteristics of solutions for these equations might be.

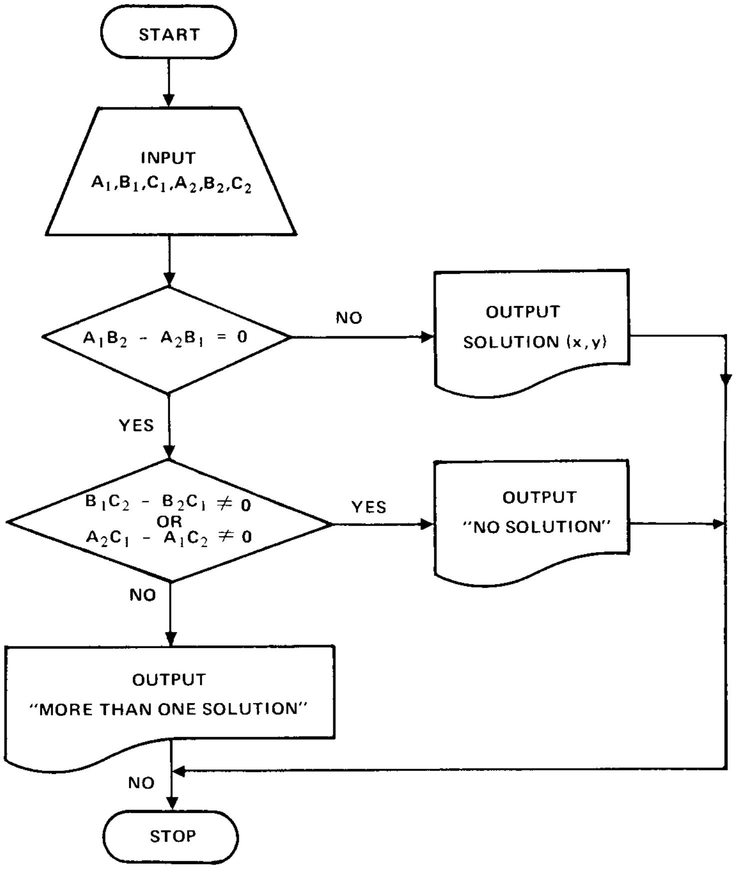
Let $N_x = B_1C_2 - B_2C_1$ $\qquad D_x = A_1B_2 - A_2B_1$

$N_y = A_2C_1 - A_1C_2$ $\qquad D_y = A_1B_2 - A_2B_1$

Then $(N_x/D_x, N_y/D_y)$ yields no solution if D_x and D_y equal zero and $N_x \neq 0$ or $N_y \neq 0$. This means the equations define parallel lines.

If D_x, D_y, N_x, and N_y all equal zero, then the system has more than one common solution and the equations define co-incident lines.

Make sure you understand why there is no unique solution in the situations described above.



Exercise 7 — Finding the Center and Radius of a Circle Determined by Three Points

Write a program that will consider three points in the Cartesian plane. If they are not collinear, find the center and radius of the circle determined. Apply the program to:

a. (-2,3); (7,6); (5,-4)

b. (6,1); (-2,1); (4,3)

c. (7,2); (-3,-5); (7,-1)

d. (3,1); (3,-1); (3,6)

Problem Analysis

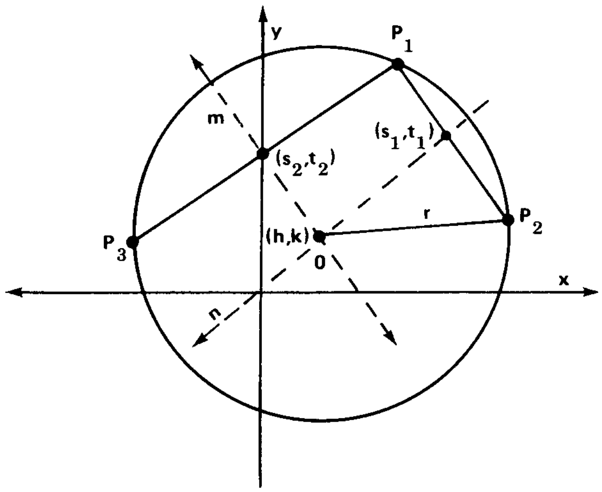


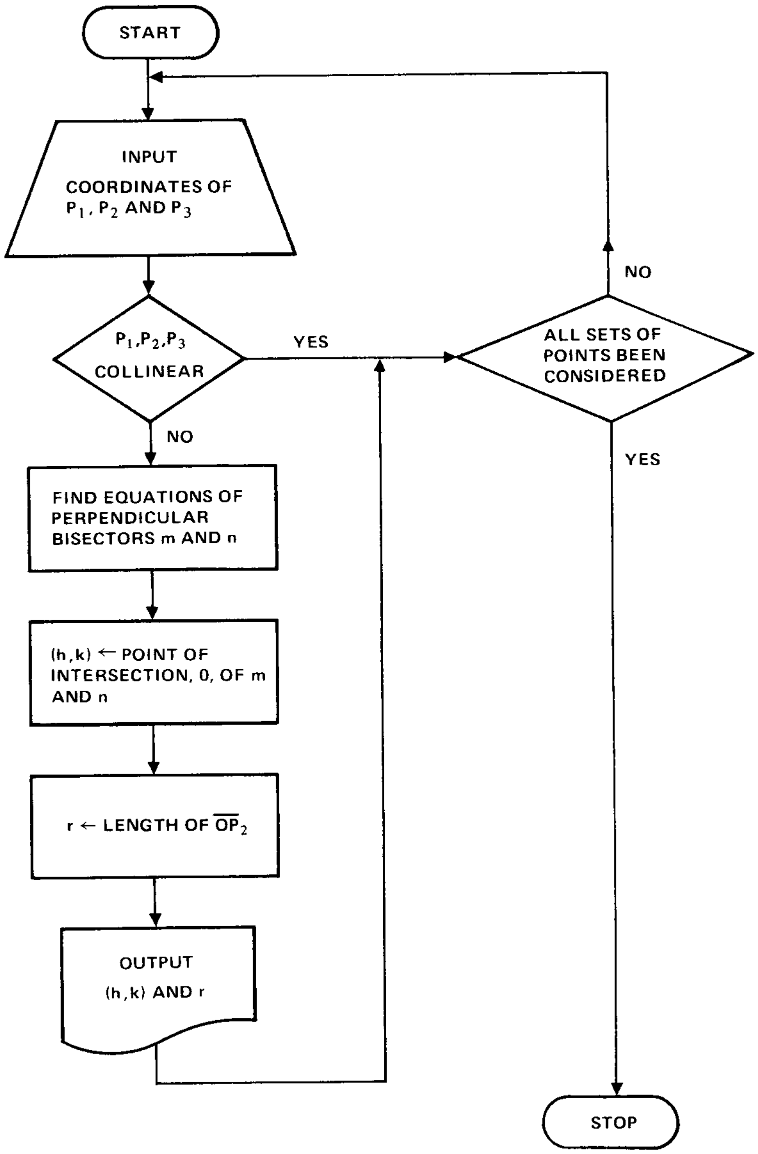
Figure 6. Graph of a Circle Determined by Three Given Points

This problem is solved by applying the techniques we have developed in the previous exercises. Figure 6 is a graph of the circle determined by points P_1, P_2, and P_3, with some dotted lines added to help us visualize the tasks to be accomplished. The problem is solved by completing the following three tasks:

(a) Find the equations of lines m and n, which are the perpendicular bisectors of line segments $\overline{P_1 P_2}$ and $\overline{P_1 P_3}$.

(b) Find the coordinates (h,k) of the point of intersection, 0, of lines m and n. This point of intersection is also the center of the circle.

(c) Find the length, r, of segment $\overline{0P_2}$. This length is the radius of the circle.


START
INPUT
COORDINATES OF
P1, P2 AND P3
P1,P2,P3
COLLINEAR
YES
NO
ALL SETS OF
POINTS BEEN
CONSIDERED
NO
YES
FIND EQUATIONS OF
PERPENDICULAR
BISECTORS m AND n
(h,k) ← POINT OF
INTERSECTION, O, OF m
AND n
r ← LENGTH OF OP2
OUTPUT
(h,k) AND r
STOP

SUGGESTED REFERENCES FOR THIS SECTION

Allendoerfer, Carl B., and Oakley, Cletus O., *Principles of Mathematics*, 2nd Edition, McGraw-Hill, New York, 1963.

Dolciani, Mary P., et al., *Algebra and Trigonometry, Modern School Mathematics*, Houghton Mifflin Co., Boston, 1968.

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Keedy, M. L., et al., *Algebra and Trigonometry*, Holt, Rinehart and Winston, New York, 1967.

Algorithms, Computation and Mathematic, Revised Edition, School Mathematics Study Group, 1966.

Computer Oriented Mathematics, An Introduction For Teachers, NCTM, 1965.

A Second Course in Algebra and Trigonometry with Computer Programming, Colorado Schools Computing Science Curriculum Development Project, 1969.

LINEAR SYSTEMS

A linear system is a set of linear equations consisting of two or more equations with the same number of variables. The number of equations in the system is called the order of the system. The following system is an example of a third order system:

$$\begin{cases} 2x + 3y - z + 2 = 0 \\ -x - 2y + 3z + 3 = 0 \\ 3x + y - 2z - 5 = 0 \end{cases} \tag{10}$$

To solve a system of equations we need to find a solution that will satisfy all equations in the system. The solution of the above system, if it exists, is a triplet (x,y,z), where x,y and z ϵ R.

Many problems in science, engineering, economics, etc., require solving systems of equations. For example, some of the systems encountered in firing guided missles or designing atomic reactors involve more than 10,000 variables and equations.

In Exercise 6 we solved a second order system, using what is known as a "Closed Form" solution. By this we mean a solution obtained by a finite number of operations. You will remember that the form developed for a second order system is quite complicated, so imagine how complex a closed form for solving a 1000th order system would be. Even closed forms for 3rd and 4th order systems are unwieldy. In this section, we will learn several methods for solving higher order systems.

Iterative Method

The iterative method described below gives us an approximate solution to a linear system, which is adequate for most real life problems. Actually, in most problems the equations themselves are approximations since the coefficients are approximate readings from instruments.

We'll illustrate this iterative method by solving the following system:

$$\begin{cases} 3x + 7y = 21 \quad \rightarrow \quad f(x) = -{}^{3}/{}_{7}x + 3 \quad \text{or} \quad f(y) = -{}^{7}/{}_{3}y + 7 \\ x + y = 2 \quad \rightarrow \quad g(x) = -x + 2 \quad \text{or} \quad g(y) = -y + 2 \end{cases} \tag{11}$$

We begin by making an educated guess at the solution of the system. The actual solution is an ordered pair (x_0,y_0). Figure 7 is a rough graph of our system.

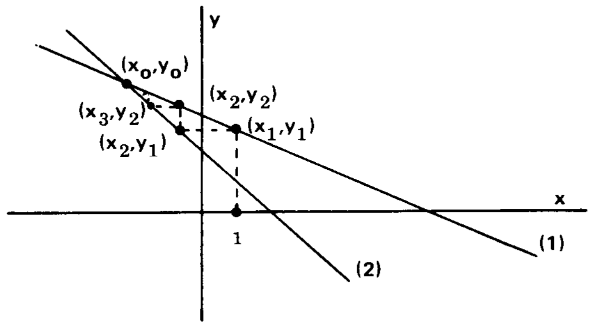


Figure 7. Graph of the Linear System

The point (x_1,y_1) is our rough guess at the solution. In this case, we said that $x_1 = 1$. Substituting x_1 into the first equation, we come up with $y_1 = -3/7\,(1) + 3 = 18/7 = 2.571$. Next we substitute y_1 into the second equation to find $x_2 = 2 - 18/7 = -.571$. This iterative process is continued in the table below. Notice that the differences in successive values of x and in successive values of y decrease as the approximations approach the actual solution.

n =	1	2	3	4	5	6	7	8	9
$x_n = 2-y_{n-1}$	*1.0	-.571	-1.245	-1.534	-1.657	-1.710	-1.733	-1.743	-1.747
$y_n = 3/7x_{n+3}$	2.571	3.245	3.534	3.657	3.710	3.733	3.743	3.747	3.749

*x_1 is our initial guess, not a calculated value

In other words, as illustrated in Figure 7, each successive point determined by the iterative process is closer to the solution (x_0,y_0). Had we continued the process we would have found the actual solution to be $(x_{17},y_{17}) = (-1.75, 3.75)$. Our example thus has a rational solution, but the solution to many systems is a pair of irrational numbers. In that case, we would determine the amount of error acceptable and continue the iteration until a solution was found within the acceptable range.

For the exercises in this unit, we will accept (x_n,y_n) as an acceptable approximation of the actual solution (x_o,y_o) when $|x_n - x_{n-1}| < \epsilon$ and $|y_n - y_{n-1}| < \epsilon$ for some predetermined small value of ϵ.

In order for the iterative process to work, you must be careful to substitute the initial guess, x_1, into the equation whose slope has the smallest absolute value. That is, if m_1 and m_2 are the slopes of two linear equations (1) and (2) respectively, so that $|m_1| < |m_2|$ then the first guess x_1 for x_o is substituted into equation (1) to find y_1. You will understand why if you draw sets of intersecting lines with both equations having positive slope, with both equations having negative slopes, and with each equation having a slope of different signs. Sketch in the successive (x_i,y_i) approximations to (x_o,y_o) when the first substitution is made into the line with the greatest absolute value of its slope. Do you see why it makes a difference into which equation you substitute your first guess?

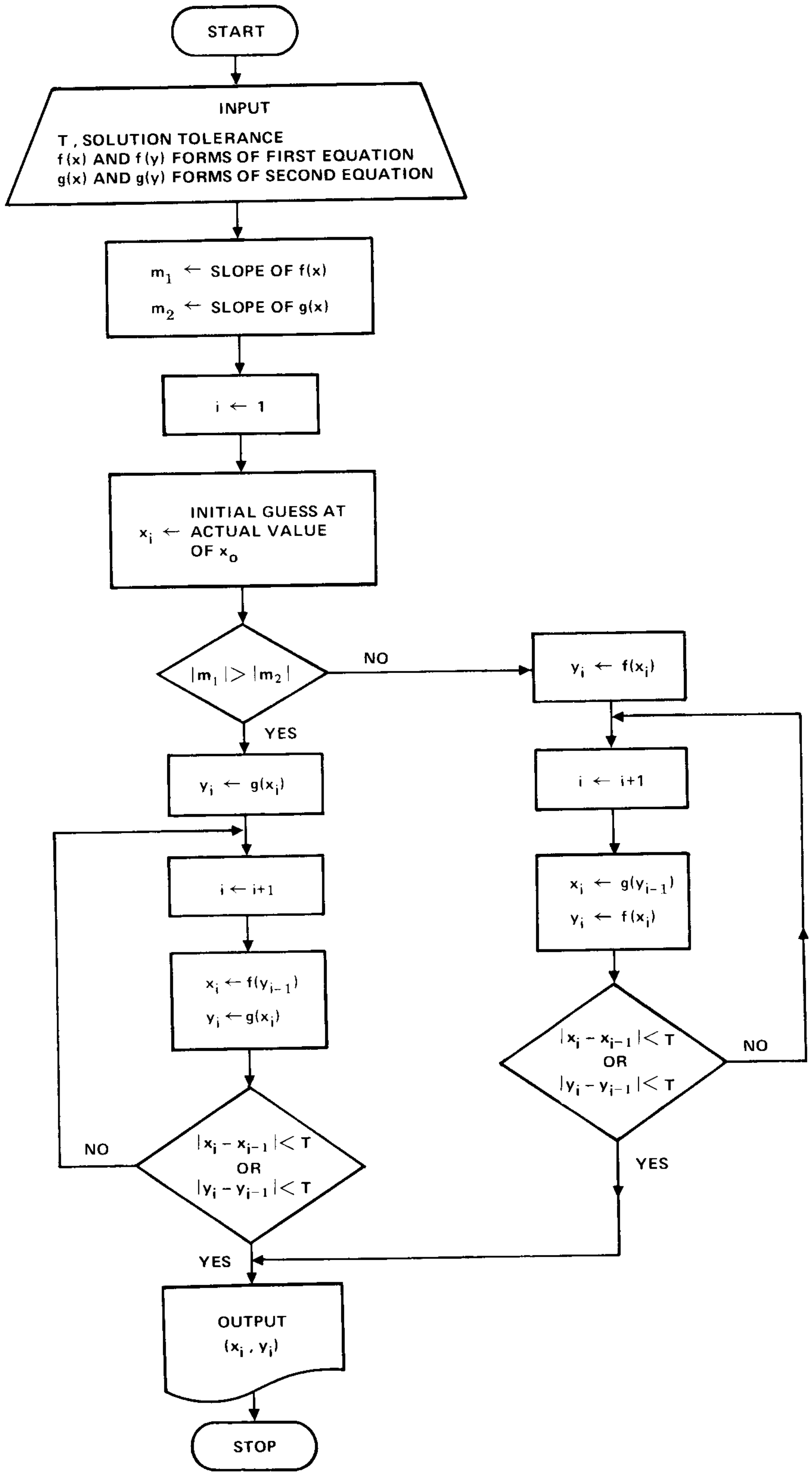
Exercise 8 — The Iterative Method

Write a computer program which uses the iterative method just described to solve the linear systems listed below. Find solutions within a tolerance of = .0001.

(a) $\begin{cases} 39x + y - 6 = 0 \\ 25x + 22y - 13 = 0 \end{cases}$ (b) $\begin{cases} 2.1356x + 86.93y + .7394 = 0 \\ .8113x + 4.793y - 6.843 = 0 \end{cases}$

Problem Analysis

This problem is a straightforward example of the process described above for solving a 2nd order system of equations.


START
INPUT
T, SOLUTION TOLERANCE
f(x) AND f(y) FORMS OF FIRST EQUATION
g(x) AND g(y) FORMS OF SECOND EQUATION
$m_1 \leftarrow$ SLOPE OF f(x)
$m_2 \leftarrow$ SLOPE OF g(x)
$i \leftarrow 1$
$x_i \leftarrow$ INITIAL GUESS AT ACTUAL VALUE OF x_o
$|m_1| > |m_2|$
NO
YES
$y_i \leftarrow g(x_i)$
$i \leftarrow i+1$
$x_i \leftarrow f(y_{i-1})$
$y_i \leftarrow g(x_i)$
$|x_i - x_{i-1}| < T$ OR $|y_i - y_{i-1}| < T$
NO
YES
$y_i \leftarrow f(x_i)$
$i \leftarrow i+1$
$x_i \leftarrow g(y_{i-1})$
$y_i \leftarrow f(x_i)$
$|x_i - x_{i-1}| < T$ OR $|y_i - y_{i-1}| < T$
NO
YES
OUTPUT
(x_i, y_i)
STOP

Gaussian Algorithm

The next method for solving systems of equations involves the use of a procedure known as the Gaussian Algorithm. The method is based on what is referred to as the "elimination" or "addition" method in some elementary algebra books. For example, the following system can be solved with this method:

$$\begin{cases} (1) \quad 2x - 3y = 2 \\ (2) \quad 3x + 4y = 6 \end{cases}$$

If we multiply (1) by 3 and (2) by -2 we arrive at an equivalent system

$$\begin{cases} (1) \quad 6x - 9y = 6 \\ (2) \quad -6x - 8y = -12. \end{cases}$$

Replacing (2) with the sum of (1) and (2):

$$\begin{cases} (1) \quad 6x - 9y = 6 \\ (2) \quad - 17y = -6 \end{cases}$$

Multiplying (2) by $-\,^1/17$:

$$\begin{cases} (1) \quad 6x - 9y = 6 \\ (2) \quad y = {}^6/17. \end{cases}$$

At this point we know that the second element, y, of our solution is $^6/17$. Substituting into (1) we find $x = {}^{26}/17$. Therefore the solution is $({}^{26}/17, {}^6/17)$.

The term "elimination" comes from the fact that in one of the equations we eliminate one of the variable terms. The x term was eliminated as we replaced the second equation with the sum of the two equations. The term "Addition or Subtraction" comes from the fact that we added two equations to get a new equation with which to replace equation (2). At any rate, this is a simple example of the use of the Gaussian Algorithm.

Before we can apply this algorithm to higher order systems of equations, we need to make a generalized statement of the procedures used above.

The following operations can be performed to transform a system of equations into an equivalent system.

a. Multiplying any equation in the system by a non-zero constant.

b. Replacing any equation in the system by an equation which is the sum or difference of two equations in the system.

c. Interchanging positions of equations within the system.

The following third order system can be solved by applying these three operations:

$$\left\{\begin{array}{llcrcrcr} (1) & 2x & + & 3y & - & z & = & 2 \\ (2) & -x & - & 2y & + & 2z & = & -3 \\ (3) & 3x & + & {}^{3}/_{2}y & - & 2z & = & 1 \end{array}\right.$$

Step 1: Multiply (1) by ${}^{1}/_{2}$.

$$\left\{\begin{array}{llcrcrcr} (1) & x & + & {}^{3}/_{2}y & - & {}^{1}/_{2}z & = & 1 \\ (2) & -x & - & 2y & + & 2z & = & -3 \\ (3) & 3x & + & {}^{3}/_{2}y & - & 2z & = & 1 \end{array}\right.$$

Step 2: Replace (2) with the sum of (1) and (2).

$$\left\{\begin{array}{llcrcrcr} (1) & x & + & {}^{3}/_{2}y & - & {}^{1}/_{2}z & = & 1 \\ (2) & & - & {}^{1}/_{2}y & + & {}^{3}/_{2}z & = & -2 \\ (3) & 3x & + & {}^{3}/_{2}y & - & 2z & = & 1 \end{array}\right.$$

Step 3: Replace (3) with the sum of (3) and -3 times (1).

$$\begin{cases} (1) & x + \frac{3}{2}y - \frac{1}{2}z = 1 \\ (2) & -\frac{1}{2}y + \frac{3}{2}z = -2 \\ (3) & -3y - \frac{1}{2}z = -2 \end{cases}$$

Step 4: Multiply (2) by -2.

$$\begin{cases} (1) & x + \frac{3}{2}y - \frac{1}{2}z = 1 \\ (2) & +y - 3z = 4 \\ (3) & -3y - \frac{1}{2}z = -2 \end{cases}$$

Step 5: Replace (3) with the sum of (3) and 3 times (2).

$$\begin{cases} (1) & x + \frac{3}{2}y - \frac{1}{2}z = 1 \\ (2) & y - 3z = 4 \\ (3) & -\frac{19}{2}z = 10 \end{cases}$$

Step 6: Multiply (3) by $-\frac{2}{19}$.

$$\begin{cases} (1) & x + \frac{3}{2}y - \frac{1}{2}z = 1 \\ (2) & y - 3z = 4 \\ (3) & z = -\frac{20}{19} \end{cases}$$

The equivalent system shown after Step 6 is sometimes referred to as the triangular form of the system.

We now have the value of z for the desired solution. We can find the values of x and y by "back solving," that is, by substituting z into (2) to find y and then finding x by substituting y and z into (1).

Back Solution:

$$(2)\quad y - 3\left(-\frac{20}{19}\right) = 4$$

$$y = \frac{76}{19} - \frac{60}{19}$$

$$y = \frac{16}{19}$$

$$(3)\quad x + 3/2\left(\frac{16}{19}\right) - 1/2\left(-\frac{20}{19}\right) = 1$$

$$x + \frac{24}{19} + \frac{10}{19} = 1$$

$$x = \frac{19}{19} - \frac{34}{19}$$

$$x = -\frac{15}{19}$$

Therefore, the solution is $\left(\frac{-15}{19}, \frac{16}{19}, \frac{-20}{19}\right)$

The solution is not difficult to find using this algorithm for 2nd and 3rd order systems, but it's obvious that the procedure will quickly become cumbersome as we go on to higher order systems. What we need is to find a way to use the computer to perform the process for us.

Look again at the entire process, Step 1 through Step 6. The process involved a series of operations performed on the numerical coefficients. If we make an array of coefficients of the above system, we have

$$\begin{pmatrix} 2 & 3 & -1 & 2 \\ -1 & -2 & 2 & -3 \\ 3 & 3/2 & -2 & 1 \end{pmatrix}$$

which we will refer to as the *coefficient matrix*. The last column contains the constants from the right side of each equation.

Compare the following arrays with the corresponding steps above.

Step 1 $\begin{pmatrix} 1 & \frac{3}{2} & -\frac{1}{2} & 1 \\ -1 & -2 & +2 & -3 \\ 3 & \frac{3}{2} & -2 & 1 \end{pmatrix}$ Step 2 $\begin{pmatrix} 1 & \frac{3}{2} & -\frac{1}{2} & 1 \\ 0 & -\frac{1}{2} & -\frac{3}{2} & -2 \\ 3 & -\frac{3}{2} & -2 & -1 \end{pmatrix}$

Step 3 $\begin{pmatrix} 1 & \frac{3}{2} & -\frac{1}{2} & 1 \\ 0 & -\frac{1}{2} & \frac{3}{2} & -2 \\ 0 & -3 & -\frac{1}{2} & -2 \end{pmatrix}$ Step 4 $\begin{pmatrix} 1 & -\frac{3}{2} & -\frac{1}{2} & 1 \\ 0 & 1 & -3 & 4 \\ 0 & -3 & -\frac{1}{2} & -2 \end{pmatrix}$

Step 5
$$\begin{pmatrix} 1 & \frac{3}{2} & -\frac{1}{2} & 1 \\ 0 & 1 & -3 & 4 \\ 0 & 0 & -\frac{19}{2} & 10 \end{pmatrix}$$

Step 6
$$\begin{pmatrix} 1 & \frac{3}{2} & -\frac{1}{2} & 1 \\ 0 & 1 & -3 & 4 \\ 0 & 0 & 1 & -\frac{20}{19} \end{pmatrix}$$

The arrays above show that we solve an n^{th} order system by making an n by (n+1) array of the variable coefficients and the constant terms. Then by applying the transformation principles, we try to arrive at an array in which the variable coefficients on the main diagonal have been changed to 1 and all elements below that diagonal are zero (as in the array above for Step 6).

In general, given a system

$$\begin{cases} A_{11}x_1 + A_{12}x_2 + A_{13}x_3 + \cdots A_{1n}x_n = k_1 \\ A_{21}x_1 + A_{22}x_2 + A_{23}x_3 + \cdots A_{2n}x_n = k_2 \\ A_{31}x_1 + A_{32}x_2 + A_{33}x_3 + \cdots A_{3n}x_n = k_3 \\ \quad\vdots \\ A_{n1}x_1 + A_{n2}x_2 + A_{n3}x_3 + \cdots A_{nn}x_n = k_n \end{cases}$$

the coefficient matrix is

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & \cdots & A_{1n} & A_{1(n+1)} \\ A_{21} & A_{22} & A_{23} & \cdots & A_{2n} & A_{1(n+1)} \\ A_{31} & A_{32} & A_{33} & \cdots & A_{3n} & A_{3(n+1)} \\ & & \vdots & & & \\ A_{n11} & A_{n2} & A_{n3} & \cdots & A_{nn} & A_{n(n+1)} \end{pmatrix}$$

where

$$A_{1(n+1)} = K_1$$
$$A_{2(n+1)} = K_2$$
$$A_{3(n+1)} = K_3$$
$$\vdots$$
$$A_{n(n+1)} = K_n$$

We have used A_{ij} for the coefficients of the variables to identify which elements of the array are involved when one of the transformation principles is applied to a particular equation of the system. Also, note that the constants of the equations have been identified by $A_{i(n+1)}$. We used x_i's to name the variables so we can have greater than a 26th order system if needed.

The following array reflects the transformation of the above *coefficient matrix*:

$$\begin{pmatrix} 1 & T_{12} & T_{13} & \cdots & T_{1n} & T_{1(n+1)} \\ 0 & 1 & T_{23} & \cdots & T_{2n} & T_{2(n+1)} \\ 0 & 0 & 1 & \cdots & T_{3n} & T_{3(n+1)} \\ & & & \vdots & & \\ 0 & 0 & 0 & & 1 & T_{n(n+1)} \end{pmatrix}$$

The T_{ij}'s are transformed elements resulting from the application of the transformation principles.

We will go through the steps necessary to solve a general third order system:

$$\begin{cases} A_{11}x_1 + A_{12}x_2 + A_{13}x_3 = k_1 \\ A_{21}x_1 + A_{22}x_2 + A_{23}x_3 = k_2 \\ A_{31}x_1 + A_{32}x_2 + A_{33}x_3 = k_3 \end{cases}$$

First set up the array

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \end{pmatrix} \quad \text{where} \quad \begin{matrix} A_{14} = k_1 \\ A_{24} = k_2 \\ A_{34} = k_3 \end{matrix}$$

Step 1: Multiply first row by $\frac{1}{A_{11}}$, $A_{11} \neq 0$

$$\begin{pmatrix} 1 & T_{12} & T_{13} & T_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \end{pmatrix}$$

$$T_{11} = \frac{A_{11}}{A_{11}} = 1$$

$$T_{12} = \frac{A_{12}}{A_{11}}$$

$$T_{13} = \frac{A_{13}}{A_{11}}$$

$$T_{14} = \frac{A_{14}}{A_{11}}$$

Step 2: Multiply row 1 by $-A_{21}$ and add to row 2.

$$\begin{pmatrix} 1 & T_{12} & T_{13} & T_{14} \\ 0 & T_{22} & T_{23} & T_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \end{pmatrix}$$

$$T_{21} = -A_{21}(1) + A_{21} = 0$$

$$T_{22} = -A_{21}(T_{12}) + A_{22}$$

$$T_{23} = -A_{21}(T_{13}) + A_{23}$$

$$T_{24} = -A_{21}(T_{14}) + A_{24}$$

Step 3: Multiply row 1 by $-A_{31}$ and add to row 3.

$$\begin{pmatrix} 1 & T_{12} & T_{13} & T_{14} \\ 0 & T_{22} & T_{23} & T_{24} \\ 0 & T_{32} & T_{33} & T_{34} \end{pmatrix}$$

$$T_{31} = -A_{31}(1) + A_{31} = 0$$

$$T_{32} = -A_{31}(T_{12}) + A_{32}$$

$$T_{33} = -A_{31}(T_{13}) + A_{33}$$

$$T_{34} = -A_{31}(T_{14}) + A_{34}$$

Step 4: Multiply row 2 by $\frac{1}{T_{22}}$, $T_{22} \neq 0$

$$\begin{pmatrix} 1 & T_{12} & T_{13} & T_{14} \\ 0 & 1 & T'_{23} & T'_{24} \\ 0 & T_{32} & T_{33} & T_{34} \end{pmatrix} \qquad \begin{aligned} T'_{21} &= \frac{T_{21}}{T_{22}} = \frac{0}{T_{22}} = 0 \\ T'_{22} &= \frac{T_{22}}{T_{22}} = 1 \\ T'_{23} &= \frac{T_{23}}{T_{22}} \\ T'_{24} &= \frac{T_{24}}{T_{22}} \end{aligned}$$

Step 5: Multiply row 2 by $-T_{32}$ and add to row 3.

$$\begin{pmatrix} 1 & T_{12} & T_{13} & T_{14} \\ 0 & 1 & T'_{23} & T'_{24} \\ 0 & 0 & T'_{33} & T'_{34} \end{pmatrix} \qquad \begin{aligned} T'_{31} &= -T_{32}(T'_{21}) + T'_{31} = 0 \\ T'_{32} &= -T_{32}(T'_{22}) + T_{32} = 0 \\ T'_{33} &= -T_{32}(T'_{23}) + T_{33} \\ T'_{34} &= -T_{32}(T'_{24}) + T_{34} \end{aligned}$$

Step 6: Multiply row 3 by $\frac{1}{T_{33}}$, $T_{23} \neq 0$

$$\begin{pmatrix} 1 & T_{12} & T_{13} & T_{14} \\ 0 & 1 & T'_{23} & T'_{24} \\ 0 & 0 & 1 & T''_{34} \end{pmatrix} \qquad \begin{aligned} T''_{31} &= \frac{T'_{31}}{T'_{33}} = \frac{0}{T_{33}} = 0 \\ T''_{32} &= \frac{T'_{32}}{T'_{33}} = \frac{0}{T_{33}} = 0 \\ T''_{33} &= \frac{T'_{33}}{T'_{33}} = 1 \\ T''_{34} &= \frac{T'_{34}}{T'_{33}} \end{aligned}$$

In order to be able to see the "back solution" process, lets transform the array into the system of equations it represents.

$$\begin{cases} (1) & x_1 + T_{12}x_2 + T_{13}x_3 = T_{14} \\ (2) & x_2 + T'_{23}x_3 = T'_{24} \\ (3) & x_3 = T''_{34} \end{cases}$$

From (3) we see that $x_3 = T''_{34}$

Back solving, we get $x_2 = T'_{24} - T'_{23}(x_3)$

And $x_1 = T_{14} - T_{13}(x_3) - T_{12}(x_2)$

From the above, we can generalize the back solution for an n^{th} order system: Compute successively the values of the variables beginning with x_n through x_i where the i^{th} variable

$$x_i = T_{i(n+1)} - T_{in}(x_n) - T_{i(n-1)}(x_{n-1}) - \cdots - T_{i(i+1)}(x_{i+1}).$$

To computerize the process above, we will replace each T_{ij} with A_{ij}. As each transformation principle was applied above, we could have continued to call each new element value A_{ij}, but then it might not have been clear that the element values of the array were changing from step to step.

Therefore

$$x_i = A_{i(n+1)}x_{n+1} - A_{in}(x_n) - A_{i(n-1)}x_{n-1} - \cdots - A_{i(i+1)}(x_{i+1}),$$

where

$$x_{n+1} = 1.$$

Exercise 9 — Applying the Gaussian Algorithm

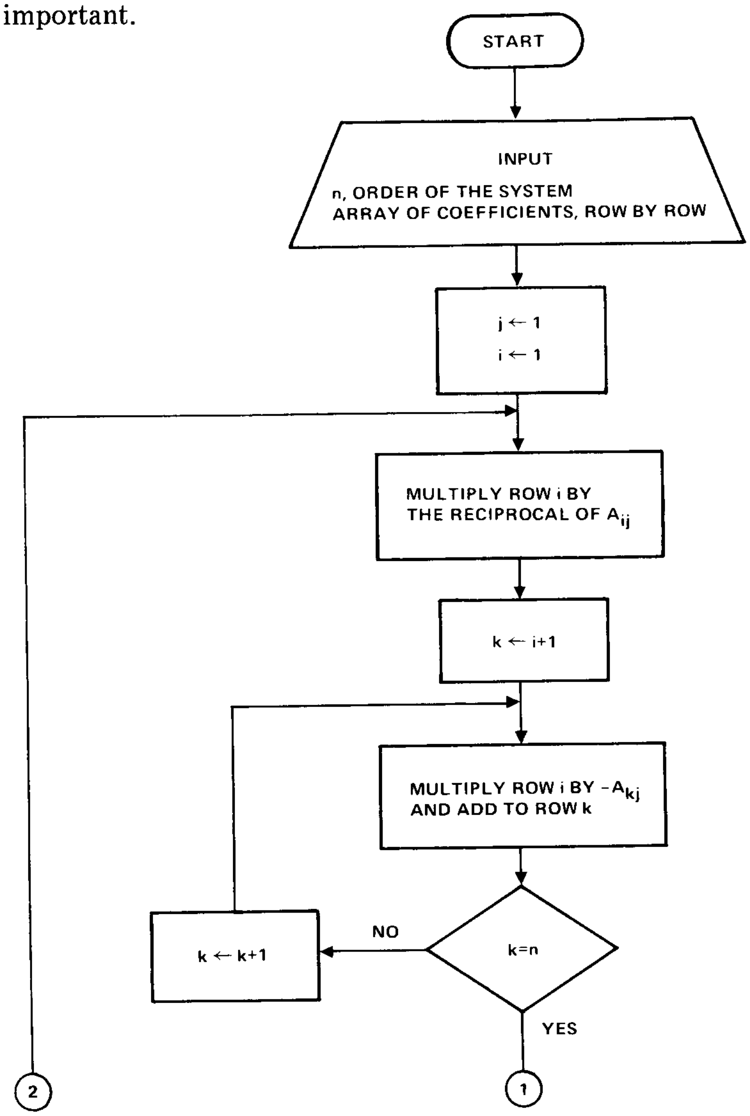
(a) Write a computer program that will solve an n^{th} order system using the Gaussian Algorithm and the back solving process described above. Apply your program to the systems of Exercise 8 and to the systems

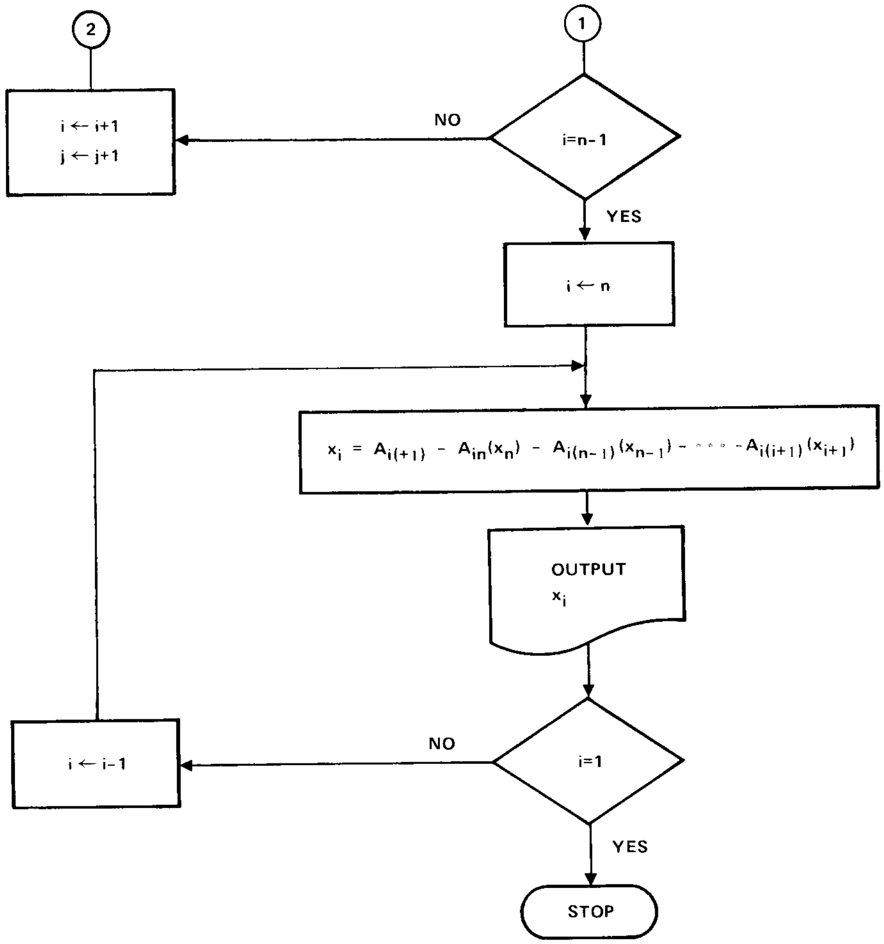
a. $$\begin{cases} 2x - y + 3z - 1 = 0 \\ -x + 3y - 2z + 0 = 0 \\ x + 1y + z + 2 = 0 \end{cases}$$

b. $$\begin{cases} 3x + 7y - 3z - w - 2 = 0 \\ 4x - 2y + z - 3w - 5 = 0 \\ -2x + 2y - 5z + 2w + 1 = 0 \\ 4x - y + 3z + w - 3 = 0 \end{cases}$$

Problem Analysis

(a) The order of performing the transformation is important, and should be done as outlined in the flowchart below. The suggested references discuss why the order is important.





(b) *Adjust the computer program for part (a) above so that it will transform the coefficient matrix into one of the form*

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \ldots & 0 & k_1 \\ 0 & 1 & 0 & 0 & \ldots & 0 & k_i \\ 0 & 0 & 1 & 0 & \ldots & 0 & k_3 \\ & & \vdots & & & & \\ 0 & 0 & 0 & 0 & \ldots & 1 & k_n \end{pmatrix}$$

Problem Analysis

(b) The desired matrix is obviously the coefficient matrix of the system:

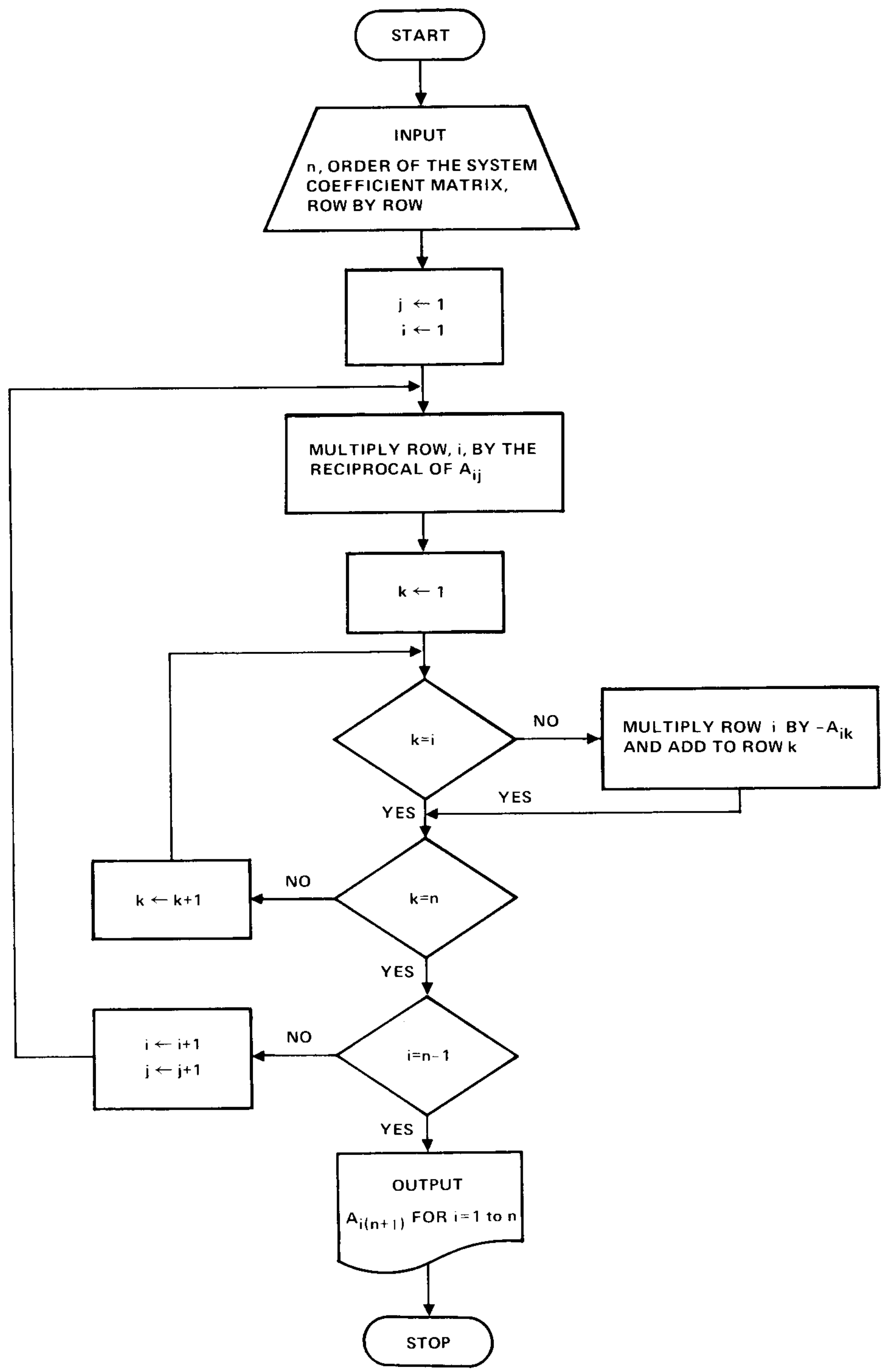
$$
\begin{array}{lllll}
x_i & & & & = k_1 \\
 & x_2 & & & = k_2 \\
 & & x_3 & & = k_3 \\
 & & & \vdots & \\
 & & & x_n & = k_n
\end{array}
$$

Therefore, given a system of equations, if we can transform its coefficient matrix to an equivalent matrix where we have the coefficients, x_{ii}, $i = 1$ to n, all equal to 1, and all other variable coefficients equal to 0, we have the solution to the system. This method of solving systems of equations, which eliminates the back solving process, is known as the *Gauss-Jordan elimination method.*

The flowchart from part (a) must be adjusted to accomplish the following:

1. Establish zero values for every element of each row, i, except the element A_{ii}, which should be equal to 1.

2. Eliminate the "back solving" loop.


START
INPUT
n, ORDER OF THE SYSTEM
COEFFICIENT MATRIX,
ROW BY ROW
j ← 1
i ← 1
MULTIPLY ROW, i, BY THE RECIPROCAL OF A_{ij}
k ← 1
k=i
NO
MULTIPLY ROW i BY $-A_{ik}$ AND ADD TO ROW k
YES
YES
k=n
NO
k ← k+1
YES
i=n-1
NO
i ← i+1
j ← j+1
YES
OUTPUT
$A_{i(n+1)}$ FOR i=1 to n
STOP

Exercise 10 — Solving Linear Systems by Use of Matrices

Write a program to solve an n^{th} order system of linear equations by the use of matrices.

Problem Analysis

Matrix algebra is a useful tool in solving linear systems of equations. If you have not studied matrix algebra, you should consult one of the suggested references before attempting this exercise.

To illustrate the use of matrices for this type of problem, consider the following system:

$$\begin{cases} 2x - 3y = 1 \\ 3x + 2y = -2 \end{cases}$$

By the definition of matrix equality, the above system can be considered as defining the equality of two 2 x 1 matrices

$$\begin{pmatrix} 2x - 3y \\ 3x + 2y \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

where (2x - 3y) and (3x + 2y) are single elements of the matrix.

By the definition of matrix multiplication, the matrix equation above can be expressed as

$$\begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

If we multiply this equation by the inverse of $\begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}$, namely $\begin{pmatrix} 2/13 & 3/13 \\ -3/13 & 2/13 \end{pmatrix}$, we end up with

$$\begin{pmatrix} 2/13 & 3/13 \\ -3/13 & 2/13 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2/13 & 3/13 \\ -3/13 & 2/13 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

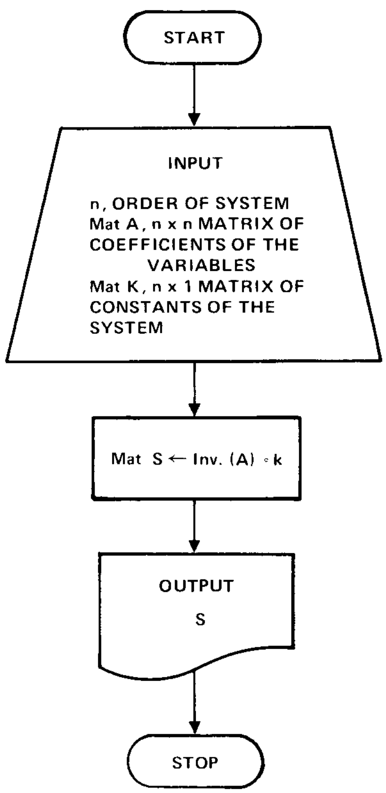
which simplifies to

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4/13 \\ -7/13 \end{pmatrix}.$$

By the definition of matrix multiplication and equality we can convert this to a system of equations

$$\begin{cases} 1x + 0y = \ \ 4/3 \\ 0x + 1y = -7/3 \end{cases}$$

which gives us the solution, $(4/3, -7/3)$, to our original system. In other words, we find the solution by multiplying the matrix of the right side by the inverse of the matrix of the coefficients.



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LINEAR PROGRAMMING

Linear programming is a technique used to determine the linear relationship among a set of variables in order to obtain the optimum of a desired objective, given a set of restraints, expressed as linear relations, on the set of variables. We will illustrate this technique by solving a simple manufacturing problem in which the optimum relation between two variables is desired, but this technique could be applied to finding a desired relation among almost an unlimited number of variables.

Linear programming is a very widely used application of computer technology. Problems demanding optimization of resources in the fields of medicine, transportation, manufacturing, farming, nutrition and many others are solved through the use of linear programming.

Exercise 11 — A Manufacturing Problem

The owner of a manufacturing firm has a slack period in his business during which part of his plant, labor and capital is not utilized to the fullest. He expects the condition to last through the two months of August and September before his regular business will again utilize all his resources.

In order to avoid laying off people and shutting down part of his plant during these two months, the owner has found a market for two types of Christmas toys which he can manufacture during this time. He can sell each type of toy in lots of 100. The profit is $150.00 per 100 on type A toys and $250.00 per 100 on type B toys. A machine which makes plastic molds, will be available. For economy, the machine must be used more than 120 hours. It can make 100 molds every 5 hours for toy A and 100 molds per 15 hours for toy B. The owner has $1300.00 capital available for this endeavor. The capital requirements are $70.00 per 100 to manufacture toy A and $50.00 per 100 for toy B. To prepare the toys for market requires two types of labor skills, assembling and packaging. During this period, 300 hours of assembling labor will be available. Toy A requires 8.0 hours per 100 assembly time and toy B, 20 hours per 100. There will be 400 hours of packaging time available. They both require 20 hours packaging time per 100 toys. How many lots of 100 of each kind of toy should be produced in order to yield the maximum profit?

Problem Analysis

The following table summarizes the resource data we are given:

Type of Resource	Total Resource Available	Resource Requirements per 100	
		Toy A	Toy B
Machine time to make molds	$\geqslant$ 120	5 hours	15 hours
Capital	$1,300.00	$70.00	$50.00
Assembly Skill	300 hours	8 hours	20 hours
Packaging Skill	400 hours	20 hours	20 hours

Now we will express the linear relationships set forth by the data. In order to maximize profit, we find a solution to Equation (1) below where A represents the number of lots of toy A to be manufactured and B the number of lots of toy B. The other equations are derived directly from the table.

(1) $150A + 250B = P$ (P, profit to be maximum)

(2) $5A + 15B \geqslant 120$

(3) $70A + 50B \leqslant 1300$

(4) $8A + 20B \leqslant 300$

(5) $20A + 20B \leqslant 400$

(6) $A \geqslant 0$

(7) $B \geqslant 0$

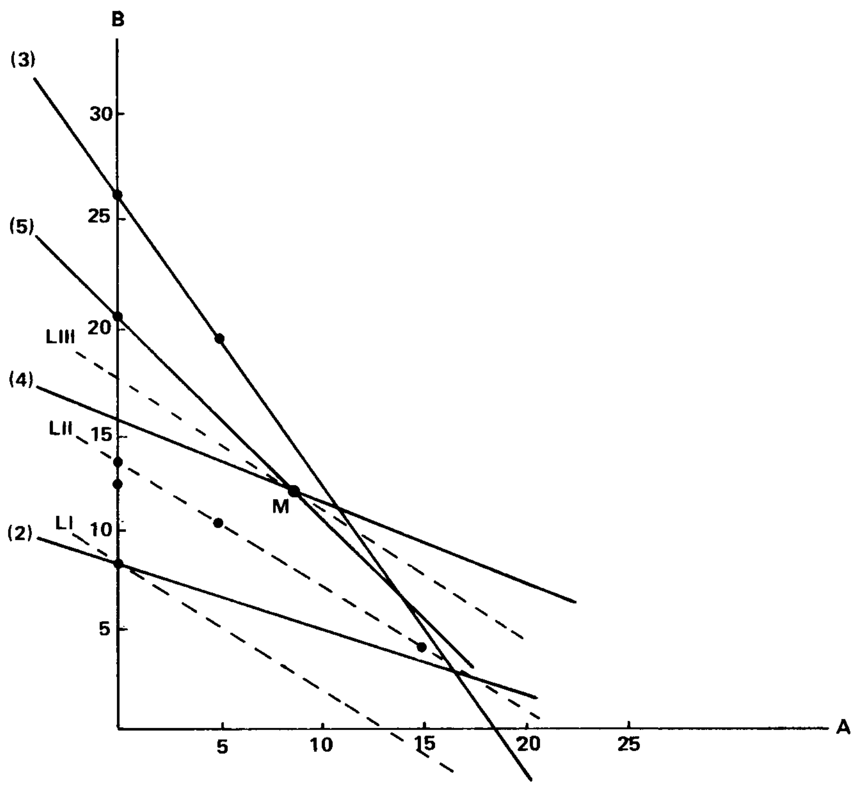


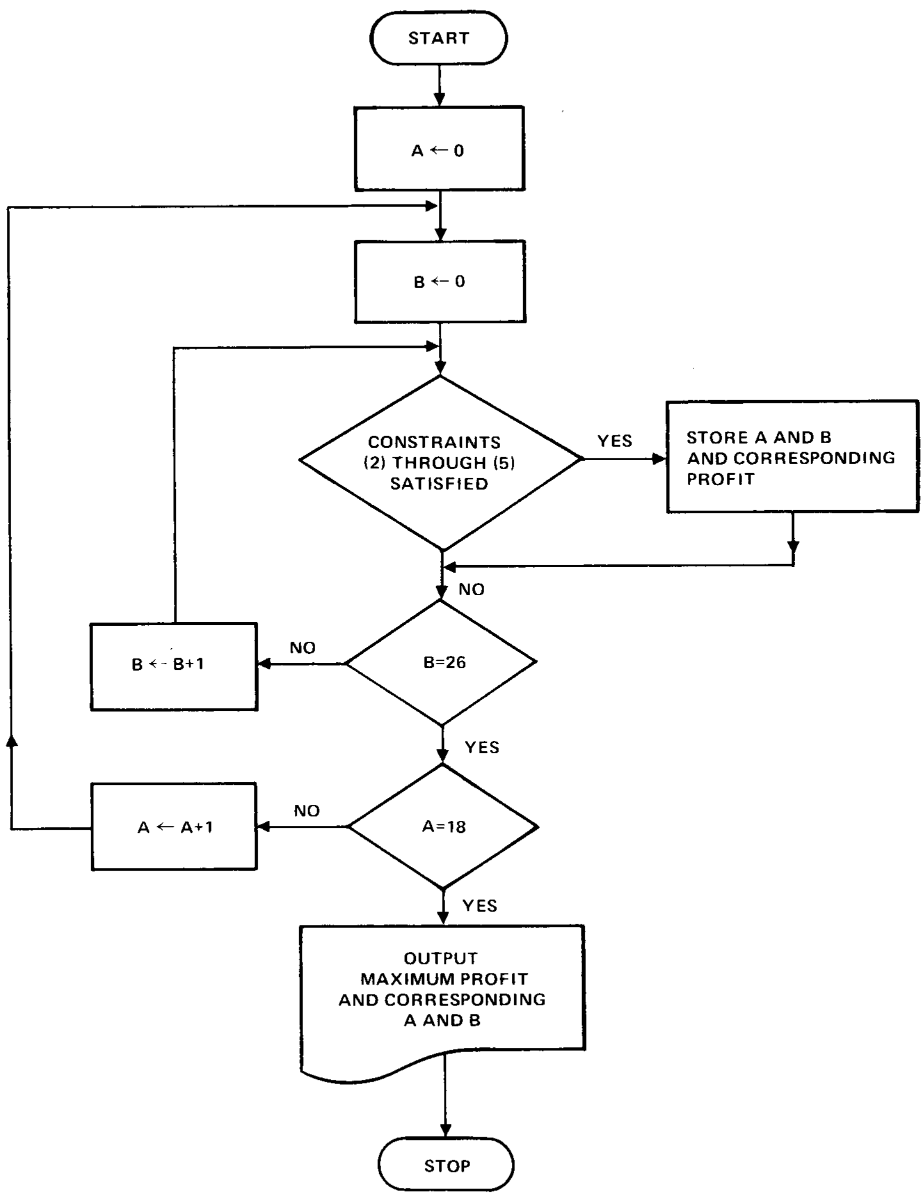
Figure 7. A Graphical Solution of Exercise 11

Equations (2) through (5) are plotted as solid lines on the graph. The profit line, Equation (1), is shown as a dotted line in three positions. Writing Equation (1) in slope-intercept form we have

$$B = \frac{-150}{250}A + \frac{P}{250} .$$

At position I, the y-intercept or $^P/250$ is 8, therefore the profit equals \$2000. In position II, the y-intercept is about 13, which makes the profit \$3250. In position III, the maximum position for the line to still intersect the profitable region, the y-intercept appears to be 17, which makes the profit \$4250. The point M seems to be the point with integral coordinates nearest the highest intersection point of the profit line. The coordinates of M are (9,11), which means 9 lots of 100 of type A toys and 11 lots of 100 of type B toys will yield the greatest profit. The actual profit should be 9(150) + 11(250) = \$4100. We will write a computer program to determine the accuracy of our graph.

In writing the program we know from Equations (6) and (7) that the minimum value of A or B is zero. From Equation (3) we can determine that the maximum values are A = 18 and B = 26. We need to consider only integer values of A and B, because one constraint was that the toys be manufactured in lots of 100.



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